The $K(\pi, 1)$ conjecture

AMSI 2026 Summer School – Student Presentations

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UQ Honours
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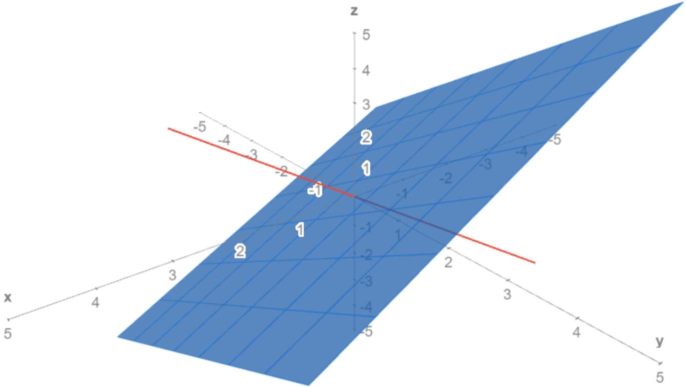
A general question

To what extent does the combinatorial data of a geometric object characterise its shape?

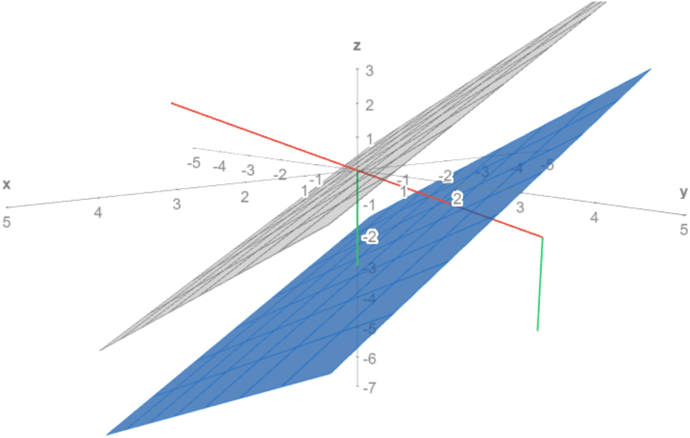
Geometry: Reflections in $\mathbb{R}^n$

A reflection in $\mathbb{R}^n$ is an invertible linear map $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ reflecting $\mathbb{R}^n$ through a hyperplane H_σ.



Geometry: Affine reflections in $\mathbb{R}^n$

A affine reflection in $\mathbb{R}^n$ is an invertible linear map $\widetilde{\sigma} : \mathbb{R}^n \to \mathbb{R}^n$ reflecting $\mathbb{R}^n$ through an affine hyperplane $H_{\widetilde{\sigma}}$.



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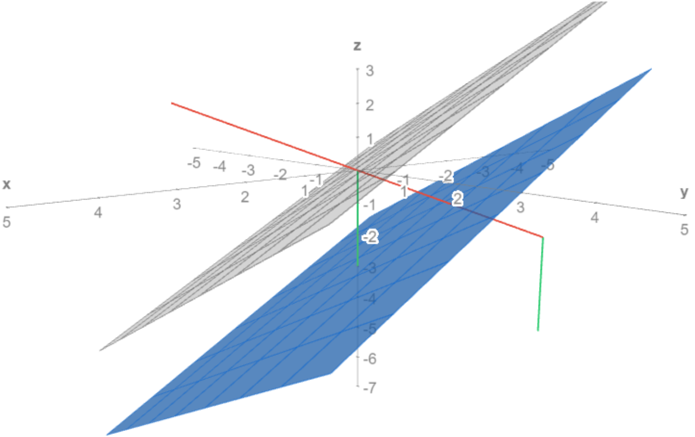

That is, $\widetilde{\sigma} = T_v \circ \sigma$ for some reflection σ and $T_v \lambda = \lambda + v$.

Geometry: Reflection groups & hyperplane arrangements

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If W is finite, $|\mathcal{A}(W)| < \infty$. If W is affine, $|\mathcal{A}(W)| = \infty$.

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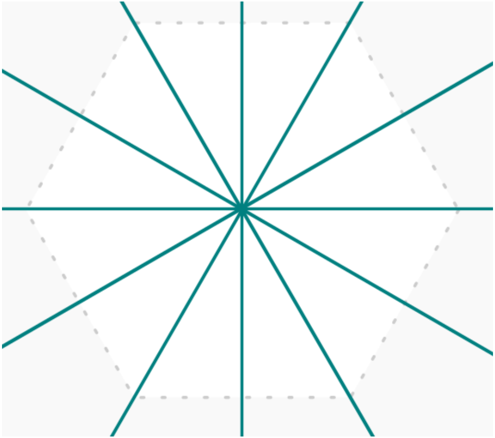
Topologically, much more interesting is

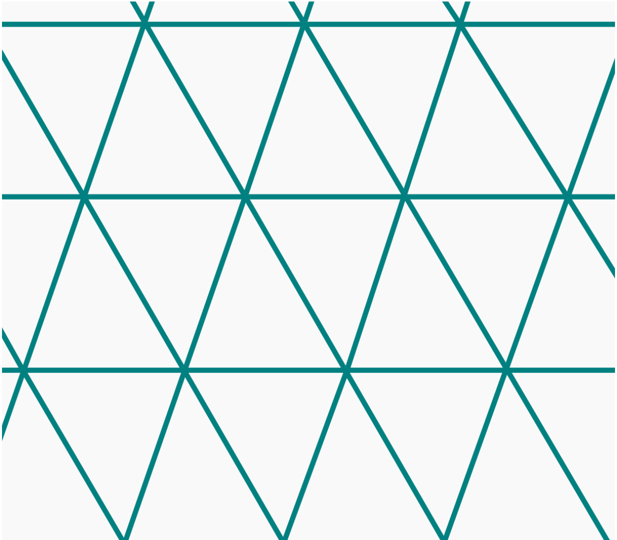
$$\mathcal{H}_W := \mathbb{C}^n - \bigcup_{H \in \mathcal{A}(W)} \mathbb{C} \otimes H.$$

called the hyperplane complement space associated with W.

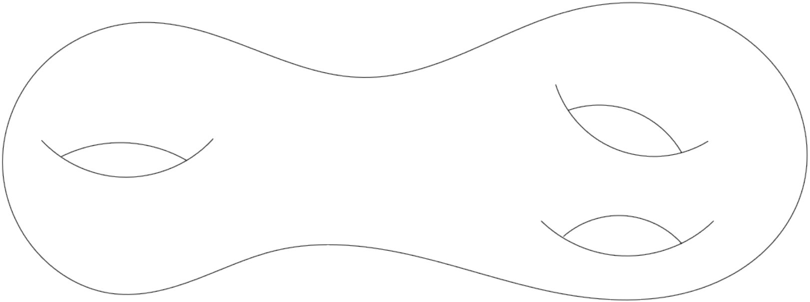
Example 1: Hyperplane complement space associated with I_6 in $\mathbb{R}^2$



Example 2: Hyperplane complement space associated with the affine symmetric group $\widetilde{A}_2$ in the UHP of $\mathbb{R}^2$



Topology: The fundamental group



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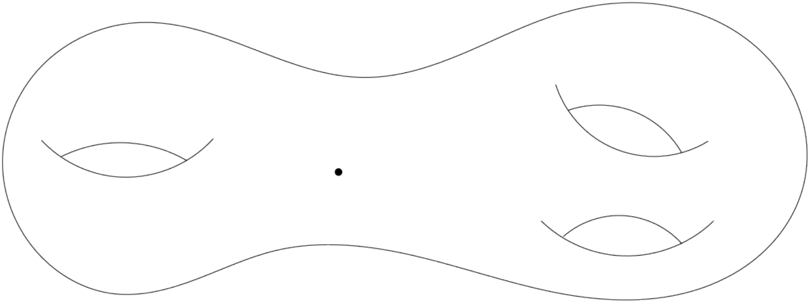

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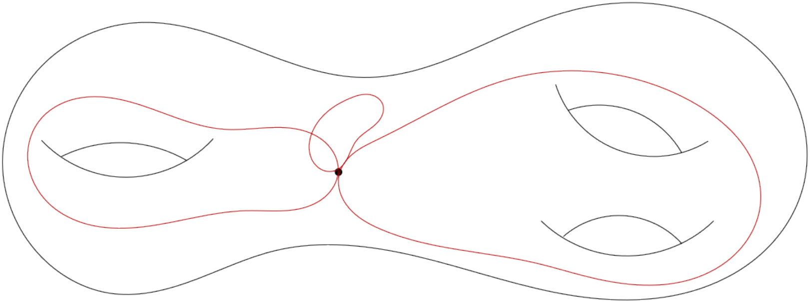

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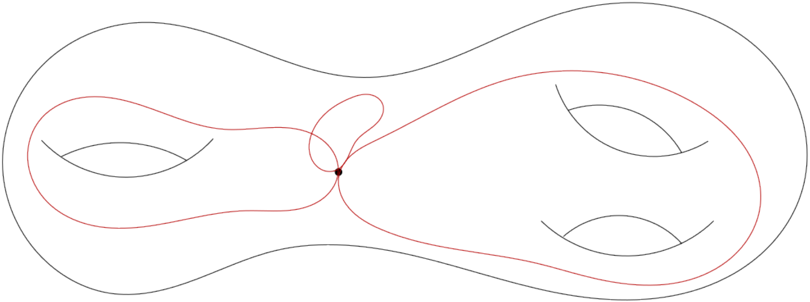

The fundamental group π_1 of the space.

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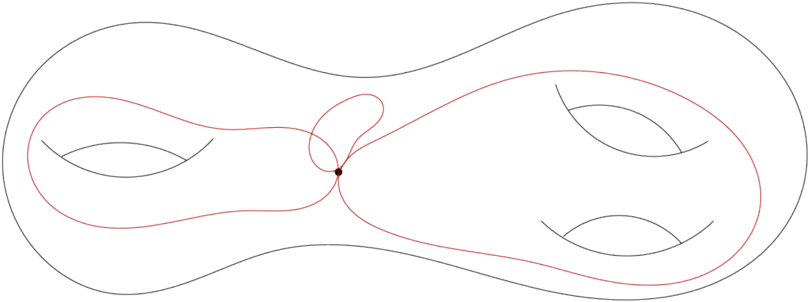

The fundamental group π_1 of the space. A topological invariant:

$$\pi_1(X) \not\cong \pi_1(Y) \implies X \text{ and } Y \text{ are topologically distinct.}$$

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$f : I^n \to X,\ f\big\rvert_{\partial I^n} = c$	$\pi_n(X)$ (nth homotopy group)

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Answer ($K(\pi, 1)$ conjecture): For each hyperplane complement space $\mathcal{H}_W$,

$$\pi_i(\mathcal{H}_W) = 1 \quad \text{for every } i \geq 2.$$

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Answer ($K(\pi, 1)$ conjecture): For each hyperplane complement space $\mathcal{H}_W$,

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That is, the homotopy of $\mathcal{H}_W$

$$\pi_1(\mathcal{H}_W), \quad \cancel{\pi_2(\mathcal{H}_W)}^{\,1}, \quad \cancel{\pi_3(\mathcal{H}_W)}^{\,1} \quad \ldots, \quad \cancel{\pi_n(\mathcal{H}_W)}^{\,1} \quad \ldots$$

is **completely determined** by π_1, the fundamental group.

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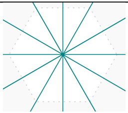
Finite		Proven	Deligne	1972

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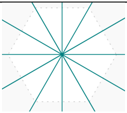
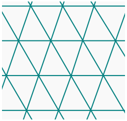
Finite		Proven	Deligne	1972
Affine		Proven	Paolini Salvetti	2021

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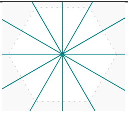
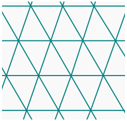
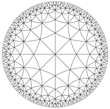
Finite		Proven	Deligne	1972
Affine		Proven	Paolini Salvetti	2021
Hyperbolic & beyond		Unresolved	-	-