The topology and geometry of Artin braid groups

Honours Project Presentation

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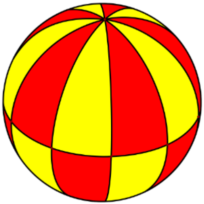
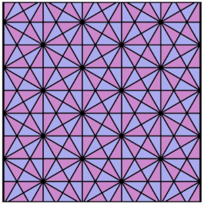
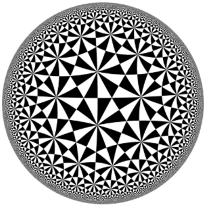
UQ Honours
Under the supervision of Masoud Kamgarpour

29 May 2026

The $K(\pi, 1)$ conjecture for Artin braid groups

- A statement about Artin braid groups and the topology of special hyperplane arrangements.

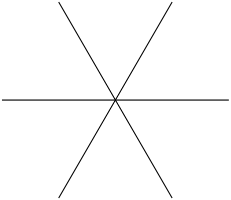


Figure: E. Brieskorn (top-left), P. Deligne (top-right), M. Salvetti & G. Paolini (bottom).

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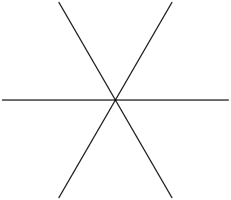

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Figure: E. Brieskorn (top-left), P. Deligne (top-right), M. Salvetti & G. Paolini (bottom).

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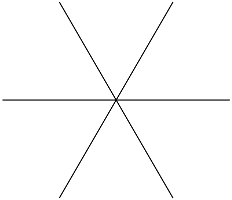

- **My project**: Understand the $K(\pi, 1)$ conjecture.
  - Outline Deligne's contribution to the problem.

Figure: E. Brieskorn (top-left), P. Deligne (top-right), M. Salvetti & G. Paolini (bottom).

Outline of presentation

Goal of talk: Introduce the $K(\pi, 1)$ conjecture.

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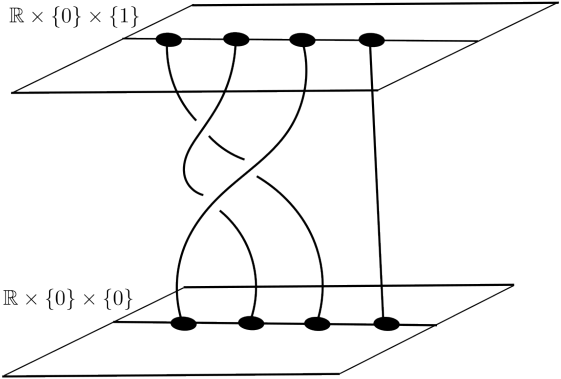
1. **Part I**: A story about the braid groups and configuration spaces.
2. **Part II**: Formulating the conjecture.

Topological braids

Consider n disjoint strands in $\mathbb{R}^3$ joining n lower points to n upper points.



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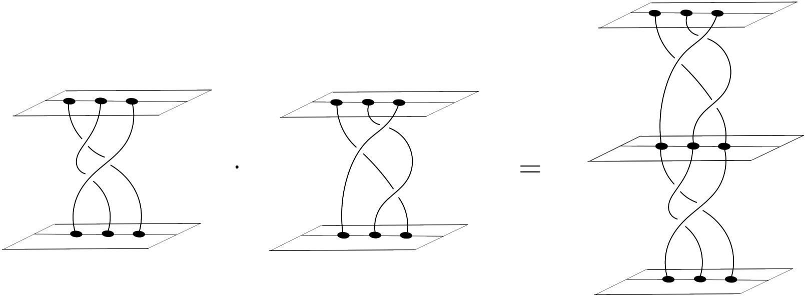

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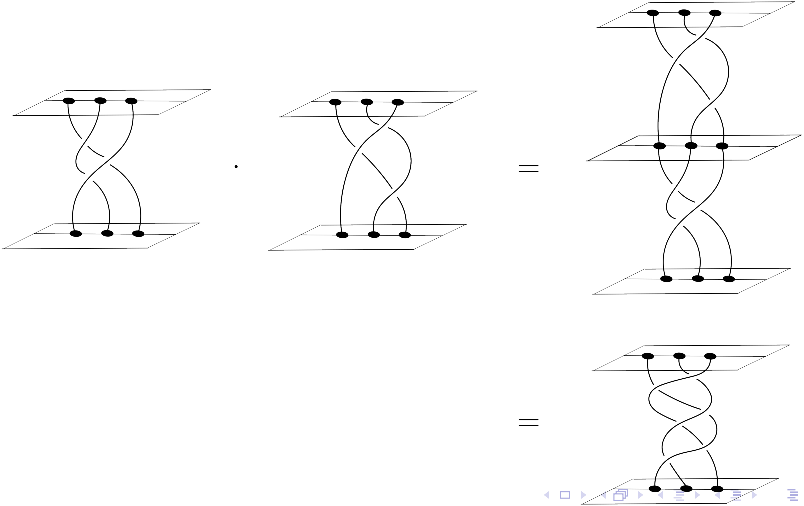
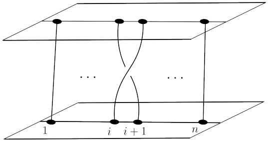
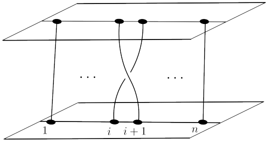


The braid group $\mathcal{B}_n$

Any braid can be decomposed into a finite product of braids of the form

$\sigma_i^+ :=$ 

, $\sigma_i^- :=$ 

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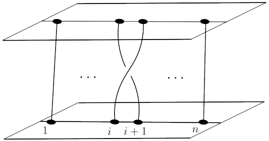
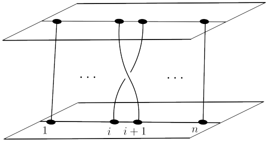
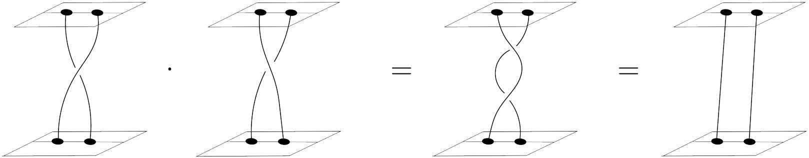
$\sigma_i^+ :=$ 

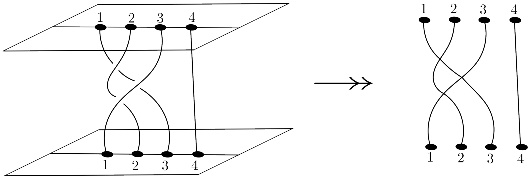
, $\sigma_i^- :=$ 

These "component" braids are inverses of each other



Connection to the symmetric group

There is a "forgetful" surjection from $\mathcal{B}_n \twoheadrightarrow S_n$



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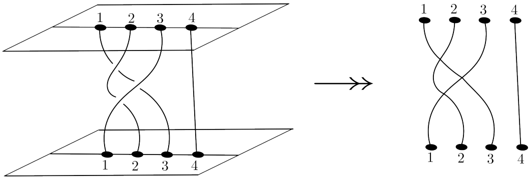
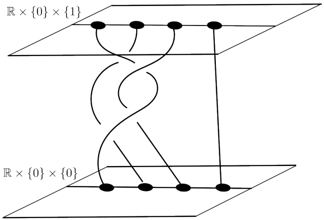


The kernel of this surjection defines the **pure braid groups** $P\mathcal{B}_n$



Configuration spaces of S_n

Let M be a topological manifold, we define the n-**configuration space** to be

$$\mathscr{C}_n(M) := M^n - \bigcup_{i \neq j} \{(x_1, \dots, x_n) \in M^n : x_i = x_j\}.$$

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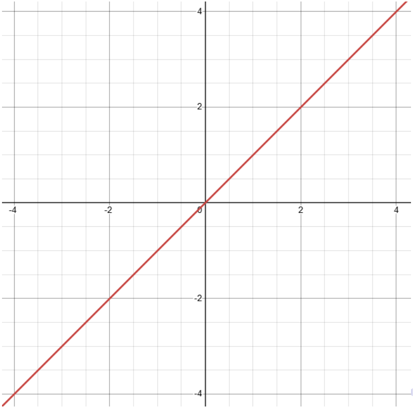
Consider $\mathscr{C}_2(\mathbb{R}) = \mathbb{R}^2 - \{(x, x) : x \in \mathbb{R}\}$:



Connection between braids and configuration spaces

A relationship between the braid groups and configuration spaces was rediscovered in the 60s by Fox and Neuwirth[1].

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$$P\mathcal{B}_n \simeq \pi_1(\mathscr{C}_n(\mathbb{C})) \quad \text{or, equivalently,} \quad \mathcal{B}_n \simeq \pi_1\left(\mathscr{C}_n(\mathbb{C}) \big/ S_n\right).$$

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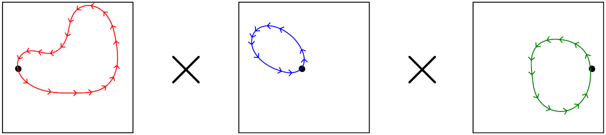
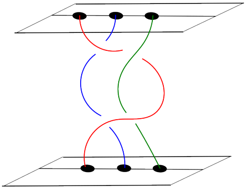
What does a loop in $\mathscr{C}_n(\mathbb{C})$ look like? Consider $\mathscr{C}_3(\mathbb{C})$:





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Higher homotopy of $\mathscr{C}_n(\mathbb{C})$

It was also observed by Fadell and Neuwirth that

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Then $\mathscr{C}_n(\mathbb{C})/S_n$ is $\mathbf{K}(\mathcal{B}_\mathbf{n}, \mathbf{1})$. This implies that

$$H^i_{\mathrm{Grp}}(\mathcal{B}_n, \mathbb{Z}) \simeq H^i_{\mathrm{Sing}}(\mathscr{C}_n(\mathbb{C})/S_n, \mathbb{Z}).$$

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The braid group has the cohomology of a manifold.

Coxeter and Artin braid systems

Given a finite set of generators $S = \{s_1, \ldots, s_n\}$:

Braid group $\mathcal{B}_n$	Symmetric group S_n
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Hyperplane arrangements of Coxeter root systems

The configuration space $\mathscr{C}_n(\mathbb{C})$ is related to the Coxeter root system Φ of S_n

$$\mathscr{C}_n(\mathbb{C}) \simeq \mathbb{C}^{n-1} - \bigcup_{\alpha\in\Phi} H_\alpha^{\mathbb{C}}$$

where H_α is a hyperplane in $\mathbb{R}^{n-1}$ determined by the root α.

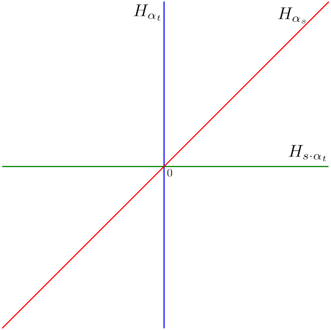

Figure: Hyperplane arrangement of S_3.

Generalising the configuration space

We can define an analogue of the configuration spaces for a finite Coxeter group W. Namely

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A similar construction exists when W is infinite, using the **Tits cone**.

The $K(\pi, 1)$ conjecture

Conjecture. Let W be a Coxeter group. Then

$$B_W \simeq \pi_1\left({}^{Y_W}/_{W}\right) \quad \text{and} \quad \pi_i\left({}^{Y_W}/_{W}\right) \simeq 1$$

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Progress on the problem

Conjecture Part I. Let W be a Coxeter group. Then

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Type	Illustration	Author(s)	Date	Status

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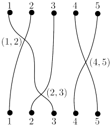
$W = S_n$		Fox & Neuwrith	1962	Solved

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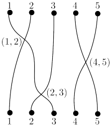
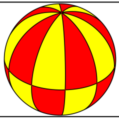
$W = S_n$		Fox & Neuwrith	1962	Solved
W finite		Brieskorn	1971	Solved

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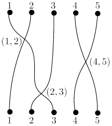
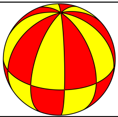
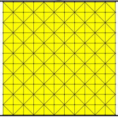
$W = S_n$		Fox & Neuwrith	1962	Solved
W finite		Brieskorn	1971	Solved
Any W		van der Lek	1983	Solved

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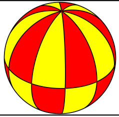
W finite		Deligne	1972	Solved

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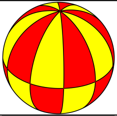
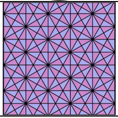
W finite		Deligne	1972	Solved
W affine		Paolini & Salvetti	2021	Solved

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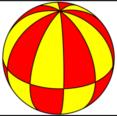
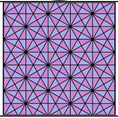
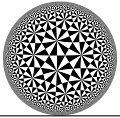
W finite		Deligne	1972	Solved
W affine		Paolini & Salvetti	2021	Solved
W hyperbolic		-	-	Open